Interactive worksheets assisting students' functional thinking conceptions in lower secondary education

EDITH LINDENBAUER, LINZ

***Zusammenfassung**: Für die vorliegende qualitative Studie wurden interaktive Arbeitsblätter entwickelt und im Rahmen einer Intervention zur Förderung funktionalen Denkens in einer 7. Klasse in Österreich eingesetzt. Diese Materialien basieren auf typischen Problemen zu funktionalem Denken und visualisieren den Darstellungswechsel zwischen situativer Darstellung und Funktionsgraph. Die Datenerhebung stützte sich auf diagnostische Tests und Interviews sowie Beobachtungsdaten. Die Forschungsinteressen umfassen insbesondere die intuitiven Vorstellungen der Lernenden sowie die Frage, ob und auf welche Weise dynamische Materialien diese Vorstellungen beeinflussen können. In diesem Beitrag wird ein Überblick über das Forschungsprojekt gegeben sowie einige Ergebnisse vorgestellt.*

***Abstract:** This paper reports on a research project about using interactive worksheets designed based on typical student difficulties concerning functional thinking. The dynamic materials focus on the representational transfer between iconic situational and graphical representation and were integrated in an intervention with a 7th grade Austrian secondary school class to foster functional thinking. Several types of data were collected through diagnostic tests, diagnostic interviews, and observations during the intervention. The qualitative study particularly pays attention to the intuitive conceptions of students and whether and in what ways interactive worksheets may influence students' conceptions. In this paper, a general overview of the research project as well as some key findings are presented.*

1. Introduction

The concept of function is an essential element of mathematics and functional thinking an important idea in mathematics education. Numerous researchers have widely investigated students' problems and conceptions in the field of functions (see section 2.2). In addition, the development of new technologies assisting the learning of functions resulted in the emergence of new aspects related to functional thinking. Dynamic mathematics software and resources based on this technology offer new opportunities for multiple, dynamically linked representations of functions. Therefore, it is important to examine how such dynamic representations influence students' learning and conceptions related to functions.

In this research project, I developed interactive worksheets applying the dynamic mathematics software GeoGebra and integrated them into a qualitative study to investigate this question. These materials were designed based on typical students' problems and misconceptions outlined in literature. They address students' conceptions and focus on the representational transfer between iconic situational model and graphical representation.

This paper outlines a dissertation project (Lindenbauer, 2018). It provides a general and detailed overview of the theoretical background and the methodological considerations. In addition, it introduces the designed interactive worksheets and summarizes some key findings. Detailed results should be presented in upcoming papers.

2. Theoretical background

The following sections outline the theoretical background of the presented research project and discuss functional thinking, student difficulties and conceptions concerning functions, and technology-related issues.

2.1 Functional thinking

Functional thinking is an important concept in mathematics education. Already over a century ago, in the reform proposals of Merano, for the first time the education for the habit of functional thinking was made explicit as a special task (Gutzmer, 1908, p. 104). Functional thinking in this context is an open and vague term, rarely attempted to be defined at the beginning of the 20th century. In the sense of functional thinking in Merano's reform, the idea of functional thinking is supposed to be a thinking habit that aimed to "penetrate" and be central of mathematics education and not only discussed as a part of function theory (Krüger, 2000).

According to Vollrath (1989), so far there exist only few attempts to define the term functional thinking because obviously it is considered to be a meaningful and simple idea and it was only seldom that people find it necessary to define. Later, he describes functional thinking as a typical way of thinking when dealing with functions (Vollrath, 1989). In contrast to the general notion of functional thinking in the reform of Merano, the concept of functional thinking is narrowed down by Vollrath and clearly situated within mathematics, especially related to the concept of function, instead of being treated as a general concept.

Vollrath (1989) states three aspects of functional thinking. Also, vom Hofe (2003) refers to three “Grundvorstellungen” concerning the concept of function. “Grundvorstellungen” are “mental models which are carrying the meaning of mathematical concepts or procedures” (vom Hofe, Kleine, Blum, & Pekrun, 2006, p. 142). Usually, not just one but several such mental models describe a mathematical concept. The development and connection of them are important for understanding a specific concept. Concerning the function concept, vom Hofe (2003) lists mental models addressing the notions of relation, co-variation or change, and object (object as a whole). They are corresponding to Vollrath’s (1989, pp. 8–16) following three aspects:

1) By functions one describes or creates connections between variables: one variable is then related to another so that one variable is viewed as dependent of the other.
2) By functions one understands how changes of one variable affect a dependent variable.
3) By functions one considers a given or generated context as a whole.

Malle (2000) refers to Vollrath’s (1989) work and specifies the following aspects in a slightly altered version without including the idea of Vollrath’s original third aspect (function as a whole). This perspective is similar to Confrey and Smith (1991), who view functions either as correspondence or as co-variation between two quantities.

- *Relational aspect*: Each argument x is associated with exactly one function value $f(x)$.
- *Co-variational aspect*: If the argument x is changed, the function value $f(x)$ will change in a specific way and vice versa.

In this context, the relational aspect corresponds to the first and the co-variational aspect to the second aspect mentioned by Vollrath (1989) and vom Hofe (2003). The relational aspect represents a static perspective of functional thinking whereas the co-variational aspect describes dynamic processes.

The aspects mentioned so far can be summarized within Vollrath’s three aspects of functional thinking. In the following, the first of Vollrath’s aspects is referred to as *relational aspect*, the second as *co-variational aspect*, and the third as *object aspect*.

In this research project, I follow Vollrath’s (1989) description of functional thinking. It relates to establishing basic conceptual understanding of the function concept and to students’ conceptions. The object aspect will further not be relevant for this project, as it is usually developed by older students (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Tasks explicitly addressing the object aspect – as for example outlined by Lichti and Roth (2019) – are not an integral part of the Austrian lower secondary education curriculum for grade 7 students, as they are in an early phase of learning to work with functions.

Functions and the different aspects of functional thinking can be represented in various ways. The following semiotic representations, which emphasize different aspects of functional thinking, are commonly used: verbal (as description of a situation), numeric (as table), graphic (as function graph), and algebraic (as formula or equation) (Büchter & Henn, 2010). The verbal representation can be extended to another relevant type, the situational representation, which could be either a verbal description or iconic representation of a real situation without use of mathematical symbols or structures (Bayrhuber, Leuders, Bruder, & Wirtz, 2010).

Vogel (2007) stresses that multiple representations of functions are able to represent aspects of functional thinking (relational as well as co-variational aspect) externally, and they have the potential to support students’ ability to interpret functions. However, representations have to be considered critically as they influence the way of thinking, they may constrain students’ thinking about the concepts involved and are interpreted by students according to their prior knowledge (Vosniadou & Vamvakoussi, 2006).

2.2 Conceptions and problems concerning functional thinking

Functional thinking is an important concept in mathematics education. Literature review reveals various problems and misconceptions in the field of functional thinking.

2.2.1 Conceptions

A significant part of research in mathematics education is concerned with examining students’ conceptions, focusing on “Grundvorstellungen” (or mental models), misconceptions, conceptual change, and preconceptions. Such research is established on the assumption that learners develop their own individual conceptions based on already existing knowledge and structures acquired in school or everyday life (Vollstedt, Ufer, Heinze, & Reiss, 2015), and is thus based on a constructivist perspective.

In research papers related to conceptions, various terms are used to describe students’ conceptions such as preconceptions, prior conceptions, alternative conceptions, misconceptions, ideas, or naive theories

(Limón, 2001; Smith, DiSessa, & Roschelle, 1994). Although these terms are used for different purposes, their common feature is that they are stressing the difference between acknowledged scientific or mathematical concepts and students' individual conceptions and ideas (Gurel, Eryilmaz, & McDermott, 2015). Several research results from different fields of mathematics education indicate that misconceptions can cause systematic errors by students (Vosniadou & Verschaffel, 2004).

For this research project, the terms error, conception, preconception, and misconception are relevant because this study focuses on students' conceptions in functional thinking.

Sfard (1991) defines the terms concept and conception in the following way:

> [T]he word "concept" . . . will be mentioned whenever a mathematical idea is concerned in its "official" form – as a theoretical construct within "the formal universe of ideal knowledge"; the whole cluster of internal representations and associations evoked by the concept – the concept's counterpart in the internal, subjective "universe of human knowing" – will be referred to as a "conception". (p. 3)

In this sense a concept is an officially acknowledged mathematical object whereas a conception is an individual, subjective construct within a person's mind based on that mathematical object. Related to Sfard's (1991) idea of conception, Gorodetsky, Keiny, and Hoz (1997) explain conception as a "mental structure that includes also the person's beliefs and basic presuppositions . . . developed from theoretical studies, from practice and from interactions with the world and society" (p. 424). These researchers include explicitly learner's preconceived ideas and beliefs as well as provide a brief explanation of how conceptions evolve.

Hadjidemetriou and Williams (2002) draw a distinction between errors and misconceptions. An error (or mistake) is an incorrect or inaccurate response to a question, or more detailed, utterances, facts or processes that deviate from an established norm (Hadjidemetriou & Williams, 2002; Prediger & Wittmann, 2009). Misconceptions can be the reasons for students' errors; Vosniadou and Verschaffel (2004) interpret misconceptions as a "knowledge system consisting of many different elements organized in complex ways" (p. 447). Misconceptions are erroneous conceptions or correct conceptions not appropriately used and are part of the knowledge structures of a person. They may be the result of instruction, the influence of everyday experience or an overgeneralization of a basically correct conception (Hadjidemetriou & Williams, 2002; Leinhardt, Zaslavsky, & Stein, 1990; Vosniadou & Verschaffel, 2004).

Furthermore, there is a distinction between pre- and misconceptions. Preconceptions can be described as conceptions based on intuition and everyday experience developed prior to systematic instruction whereas misconceptions are the result of instructional influence (Clement, Brown, & Zietsman, 1989; Vosniadou & Verschaffel, 2004). Synonymous to preconception, I will use the term intuitive conception to stress the influence of everyday knowledge and that students are unaware of such conceptions.

Students' misconceptions can lead to problems and learning difficulties. The next section contains a description of different students' problems in the area of functional thinking interesting for lower secondary education.

2.2.2 Student problems

For comprehensive students' understanding of the concept of function, the development and training of all three aspects of functional thinking is important. According to Busch, Barzel, and Leuders (2015b), these aspects also affect the emergence of students' misconceptions or errors. Malle (2000) states that it is important for students to develop relational and co-variational aspects already in grades 5 to 8, especially in connection with interpretations of functions in a real-world context. He considers it difficult to catch up on these mathematical skills if they are neglected in lower secondary education.

Several authors report on the dominance of the relational aspect in teaching (Confrey & Smith, 1991; Leinhardt et al., 1990; Malle, 2000; Stölting, 2008). Hoffkamp (2011), for example, states the over-emphasis of the relational aspect as related to a dominance of numerical approaches to the function concept by using tables. Also, modern definitions of the concept of function are based on the idea of correspondence and thus emphasizing the relational aspect. In summary, one could assume that the relational aspect is the least difficult to learn.

For these reasons, the situation is different for the co-variational aspect. Already Goldenberg, Lewis, and O'Keefe (1992) emphasize the problem of regarding functions dynamically especially on static media such as paper. Let us consider the following typical task for students: How does the area of a circle change when the radius is doubled? This question emphasizes the co-variational aspect of the functional dependency between radius and area of a circle. It does not seem to be difficult to solve; however, empirical results show that particularly the co-variational aspect is inaccurately or hardly developed by students although it is important to be able to work with functions in practice (De Bock, Verschaffel, &

Janssens, 1998; Malle, 2000; Hoffkamp, 2011). According to Malle (1993) the co-variational aspect is closely linked to the varying aspect of variables (Veränderlichenaspekt), which is also often hardly developed.

An underdeveloped co-variational aspect is related to commonly used representations of functions. For example, the representation as formula stresses the relational aspect of functions. Additionally, in mathematics education without use of technology, function graphs are often regarded only statically, which does not foster the dynamic view of the co-variational aspect. Leinhardt et al. (1990) suggest interpreting graphs of functions qualitatively in order to examine the co-variational aspect of functions.

Students' ability to flexibly use and change between representation registers is the foundation of mathematical understanding of functions (Duval, 2006). Similarly, other researchers emphasize the importance of the ability to switch back and forth between different representations, to identify and link the connecting elements as an evidence of conceptual understanding (Ainsworth, Bibby, & Wood, 2002; Bayrhuber et al., 2010; Leuders & Prediger, 2005).

Various students' difficulties in the context of functional relationships are related to transfers between different kinds of representations such as verbal, numeric, graphic, and algebraic. According to Duval (2006), many students' problems in mathematical understanding are due to the complexity of conversions (transformations between representations of different registers). "Changing representation registers is the threshold of mathematical comprehension for learners at each stage of the curriculum" (Duval, 2006, p. 128). Therefore, to understand functional dependencies, the representational changes are of particular importance because at least two registers are involved in such a mathematical activity (Duval, 2006; Nitsch, 2015).

Especially representational transfers between graphical representation and verbal description of a situation can be classified as particularly difficult, for example, when students struggle with difficulties due to inadequate interpretation of everyday experience. On the one hand, this is justified by the fact that a situational description contains a high number of so-called fact gaps and confounding facts; on the other hand, these representational transfers require global instead of local interpretive activities, which are more prone to errors for students (Bossé, Adu-Gyamfi, & Cheetham, 2011).

Further reasons for the particular difficulty of situational descriptions may be the influence of everyday experience on students' conceptions (preconceptions) as well as linguistic aspects. If the description of a situation is provided verbally, a certain understanding of text and further linguistic abilities are necessary for dealing with this kind of representation in problem solving processes (Prediger, 2013).

The representational transfer between graphical and situational (verbal, iconic) representation is particularly problematic; moreover, it is interesting in lower secondary education due to the prior knowledge of students. Therefore, I describe additional research results about problems related to the interpretation of graphs: graph-as-picture error and slope-height confusion.

A rudimentary developed co-variational aspect might lead, among other things, to a *graph-as-picture error*. Janvier (1981) and Clement (1985) were among the first to examine and discuss the error of treating a graph as a picture. This error occurs in various forms and means that students interpret a function graph as photographic image of a real situation instead of an abstract representation of the dependency of one quantity on another and their co-variation (Busch et al., 2015b; Clement, 1989; Hadjidemetriou & Williams, 2002; Schlöglhofer, 2000).

Graph-as-picture errors are provoked accordingly by the represented context; therefore, some situations especially lead to misinterpretations of function graphs. Especially in the context of distance-time diagrams, function graphs are interpreted frequently in such iconic ways (Leinhardt et al., 1990; Schlöglhofer, 2000).

Difficulties arise also in the interpretation of slope and growth, for example, if the point of maximum growth is confused with the largest function value. This is referred to as *slope-height confusion* (Clement, 1985, 1989). The confusion of slope and height indicates that students do not comprehend the concept of slope and that they focus on the position of the function graph instead on the slope in the respective time period (Busch, Barzel, & Leuders, 2015a).

Another problem for students is the so-called *illusion of linearity*. It means that linear or directly proportional models are preferably used for the description of relations even if they are not applicable. For example, some students' quotes from the research project of Hoffkamp (2011, pp. 107–108) refer to this misconception: ". . . the function has no slope, because if it would have a slope, it would be straight" [1] (p. 107), or "graph is always . . . [student points on straight line]" [1] (p. 108).

Van Dooren, De Bock, Janssens, and Verschaffel (2008) additionally point out that "[s]tudents can apply particular characteristics and representations without being aware that these are actually related to linearity and even without (fully) grasping the idea of linearity" (p. 315). This misconception evolves not just in the context of graphical representations of functions but also in other domains, such as non-proportional arithmetic word problems; probabilistic reasoning; number patterns, algebra, and calculus; and geometrical reasoning (Van Dooren et al., 2008).

According to De Bock, Van Dooren, Janssens, and Verschaffel (2007), this misconception is reasonably persistent, especially when it has to do with the enlargement of two-dimensional figures (e.g., students' belief that doubling the side length of a square lead also to a doubling in area), which is also related to the co-variational aspect of functional thinking. There exist different explanations for the appearance and persistence of the illusion of linearity. First, proportional reasoning seems to be an intuitive knowledge based on early and repeated everyday experiences that is resistant to change and also influencing students' preconceptions. Second, classroom experiences are also responsible for students' overuse of linear and direct proportional models as frequent experiences with the applicability of proportional models influence students' conceptions and lead to inappropriate habits and beliefs about mathematical modeling. Moreover, students' gaps in specific content knowledge (e.g., fragmentary geometrical knowledge) may also play a role in errors related to the illusion of linearity (De Bock et al., 2007; Van Dooren et al., 2008).

The outlined problems can cause students' misinterpretations of functions and especially of graphs of functions. Although many international research results discussing functional relationships exist, there are only a few studies in the German-speaking research community about students' misconceptions and little research attending to students of lower secondary school (e.g., Nitsch, 2015; Busch et al., 2015b; Bayrhuber et al., 2010). Particularly, a focus on lower secondary students' individual conceptions seems to be missing.

2.3 Use of technology

During the past three decades, the development of technology and technology-based resources started to transform mathematics teaching and learning by offering new opportunities. Although there were high hopes of mathematics education researchers in the late 1980s and early 1990s about technology integration into school education, these were not fulfilled. However, educational technology could offer new potentials for mathematics teaching and learning (Drijvers et al., 2016; Lavicza, 2010).

Borba and Confrey (1996) point out that structures, processes and objectives of mathematics and mathematics teaching change through the use of technology in mathematics education because mathematics does not exist independently of its forms of representations. Therefore, new technological environments provide new representational resources and types of representations, both with the potential to support different ways of mathematics teaching and learning (e.g., Morgan & Kynigos, 2014; Morgan, Mariotti, & Maffei, 2009).

2.3.1 Dynamic representations

Due to the technological developments, the possible semiotic representations are changing recently, these representations now also including dynamic and interdependent forms. For example, mathematics software can combine graphical and algebraic representations of functions in such a way that manipulating one representation immediately affects the other. Also, the graphics view in dynamic geometry software enables users to drag, turn, enlarge or decrease representations of mathematical objects or view them from various perspectives. In essence, new technological environments offer the possibility to examine mathematical concepts with dynamically linked, multiple representations by not only displaying these representations but also by allowing various actions on them as well as connecting and translating between different representations (Arzarello, Ferrara, & Robutti, 2012; Moreno-Armella, Hegedus, & Kaput, 2008).

Morgan and Kynigos (2014) emphasize the potential of linked multiple representations of mathematical objects in enriching students' conceptual understanding. The ways students manipulate and interact with dynamic representations may significantly differ from working with static representations in a paper-and-pencil environment. Additionally, manipulating representations influences and enriches mathematical conceptualization because these manipulations could lead to representational changes that in turn become part of these representations (Morgan et al., 2009).

Of course, paper-and-pencil environments are also able to provide representations with dynamic properties, but dynamic media could allow representations to change over time, thus facilitating variation and examination of invariants of mathematical objects (Kaput, 1992; Arzarello et al., 2012). Therefore, dynamic representations seem to be a powerful tool for examining the co-variational aspect of functional thinking. Additionally, the feature of direct manipulation of dynamic representations allows to comprehend and to examine connections between different semiotic representations as well as certain properties of the represented mathematical objects (Kieran &

Yerushalmy, 2004). Research results show the potential of dynamic representations in supporting concept development processes and fostering students' conceptual understanding (Heid & Blume, 2008; Hoyles, Noss, Vahey, & Roschelle, 2013). Lichti and Roth (2018), for example, examined whether functional thinking of sixth graders could be fostered better using real materials or computer-based simulations. The results of their pre-post-test-intervention study indicate that utilizing simulations seems to be more beneficial than working with real materials, especially regarding the co-variational aspect.

The next paragraphs will provide a description of specific features of the utilized technology as well as their educational potential.

2.3.2 Dynagraph representation

The co-variational aspect includes the dependency of one variable on another and thus the dynamic perspective on functional dependencies. At present, the Cartesian coordinate system is predominant for graphical representations of functions in mathematics teaching and learning. Another possible graphical representation is based on software called DynaGraph providing two parallel coordinate lines (with or without scaling) instead of two perpendicular axes (Goldenberg et al., 1992). In the following, I will refer to this kind of coordinate system as *dynagraph* representation.

In a Cartesian coordinate system, students may perceive graphs of functions only as static "images" which may lead to a misinterpretation of graphs as pictures while the dynamic aspect of functions moves into the background. In contrast to representations in a Cartesian coordinate system, in a dynagraph environment the independent variable *x* can actually be varied (e.g., by mouse-manipulation) and the corresponding function value *f(x)* can be observed. Furthermore, the variable and the function value are displayed separately and not fused like in Cartesian graphs (see Figure 1). Therefore, dynagraph representations enable a more profound examination of the co-variational aspect (Goldenberg et al., 1992; Malle, 2000).

According to Goldenberg et al. (1992), dynagraphs are powerful tools to examine characteristic properties of functions, to analyze functions qualitatively and therefore to support the understanding of functions. Dynagraphs are able to complement static representations by adding a dynamic perspective, thus enriching the conceptual development in this domain.

2.3.3 Dynamic mathematics software (DMS)

For this research project, I decided to use GeoGebra, an open-source mathematics software for educational purposes, because GeoGebra is the most widely employed mathematics software in Austrian schools. GeoGebra (www.geogebra.org) is a dynamic mathematics software (DMS) integrating a computer algebra system (CAS) into a dynamic geometry system (DGS). Its potential lies in the ability to combine geometry, algebra, spreadsheets, statistics, and calculus (International GeoGebra Institute, 2017).

The different windows of GeoGebra (e.g., algebra, graphics, geometry, spreadsheet) enable to connect different semiotic representations of mathematical objects. The representations of mathematical objects presented in different windows are linked dynamically in such a way that changing one representation immediately affects the other(s). This feature allows teachers and students to explore and examine multiple representations of mathematical objects (Hohenwarter & Jones, 2007).

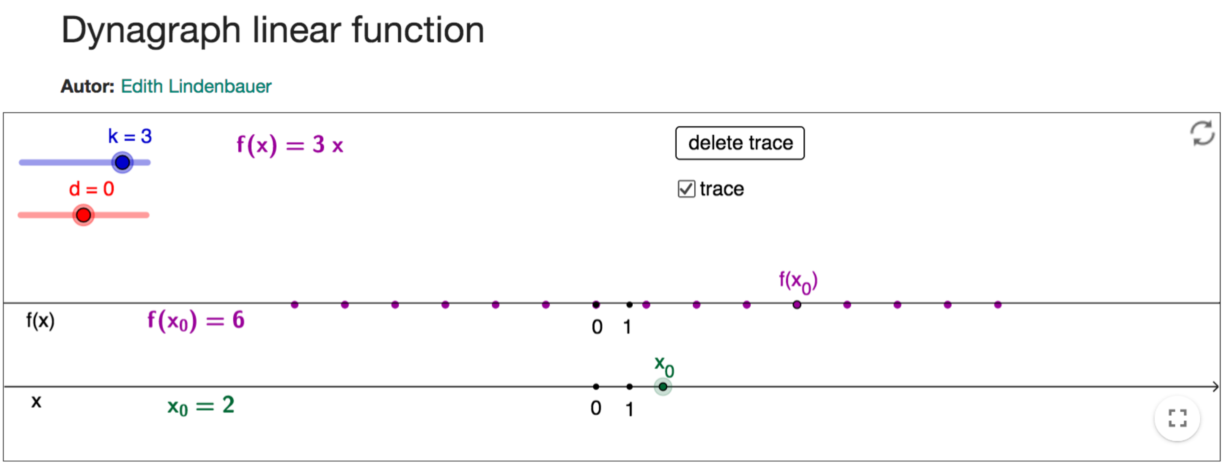

Fig. 1: Interactive worksheet "Dynagraph linear function", https://ggbm.at/navMMXAP

Misfeldt (2011) describes the potential of GeoGebra with regard to Duval's (2006) framework about semiotic registers by its feature to combine simultaneously different semiotic representations in separate windows. With respect to transfers between two representations of different registers, which are especially difficult for students to achieve, dynamically linked multiple representations could provide a cognitively different approach compared to static representations in a paper-and-pencil environment. Furthermore, the use of DMS may support students' development of functional thinking because it is suitable to emphasize relational as well as co-variational functional aspects, the latter providing a dynamic perspective on functional dependencies (Falcade, Laborde, & Mariotti, 2007).

So far, quantitative research findings about the use of technology in teaching reveal at best moderate effects on students' learning achievements (Drijvers et al., 2016). Results concerning dynamic representations appear to be more promising because these representations can support students in understanding mathematical concepts (Hoyles et al., 2013).

The question remains, why technology-based material might be supportive and what happens in students' minds when they are working with interactive dynamic representations. Therefore, we need to examine in more detail the influence of technology on students' individual conceptions and the opportunities technology may offer to support students' development of mathematical understanding.

2.3.4 Research questions

Various researchers (e.g., Leuders & Prediger, 2005; Vosniadou & Vamvakoussi, 2006) suggest introducing mathematical concepts at an earlier stage in mathematics education in order to help students in building a diverse concept image and to avoid that intuitive conceptions develop to misconceptions. This led to my research interest in students in an early phase of learning functions. In Austria, this group is represented by students of grade 7 and beginning grade 8 (age 12 to 13). Students of these grades have already learned how to interpret Cartesian coordinates; therefore, they are able to study graphical representations, but they are not accustomed to the concept of the function and have only little experience with different functional relationships.

The literature review draw attention to the following empirical research questions of the presented research project. As I am especially interested in the influence of technology-based dynamic representations on the students' conceptions, these conceptions have to be examined first. Consequently, the first research question is:

1) What different conceptions, with particular attention to pre- and misconceptions, emerge concerning functional thinking of students of lower secondary education in an early phase of learning functions (grades 7 to 8)?

The aim of the first question is to reveal the range of various conceptions of students at grade 7 concerning tasks addressing problems as outlined in section 2.2. Furthermore, I focus on preconceptions, which may develop into misconceptions, and especially different levels of conceptual understanding concerning tasks addressing graph-as-picture errors, distance-time diagrams addressing the slope-height confusion, and illusion of linearity in tasks related to the enlargement of two-dimensional figures.

So far, quantitative studies reveal at best moderate effects on students' achievement, but research concerned with dynamic representations seems to be more promising. However, we need to understand what happens with students' conceptions when they are working with this kind of technology-based material. Are such materials able to support students? And if yes, what happens in detail? As the literature review reveals a lack of qualitative research about the influence of dynamic materials on students' conceptions at this age, these considerations lead to the second research question, which seeks to investigate any influence of working with the interactive materials on students' conceptions and comprises following aspects:

2) In which ways do students of lower secondary education utilize the designed interactive materials? How do students of lower secondary education perceive and interpret information presented in the dynamic materials? What are the potentials of the interactive materials in supporting students' conceptual development regarding the presented tasks and what kind of problems (e.g., misinterpretations) could be involved?

The second question includes a broad approach for identifying main factors influencing the learning process, difficulties to be aware of (e.g., misinterpretations of presented information), and also possible potentials of dynamic materials.

In the next section, the applied dynamic materials are described in more detail.

3. Dynamic materials

These dynamic materials were designed to foster functional thinking, especially to support students in translating between situational and graphical representations. Basically, the materials are intended to be utilized in regular classroom settings, in which the materials could either be applied alone or combined together. The accompanying tasks should encourage students to explore the linked representations, more detailed to verbalize observations, to consider questions and discuss them in pairs, to make conjectures and check them, and to reason answers. In general, the tasks should encourage students to analyze the presented functional dependency first qualitatively and then quantitatively including questions addressing the relational and co-variational aspects.

A collection of original German-language interactive worksheets used during data collection in the research process can be found at https://ggbm.at/EFVg7W8V; the translated English version can be retrieved here: https://ggbm.at/ftqpETqJ.

3.1 Basic design considerations

Based on the problems and examples related to students' conceptions (see section 2.2), several interactive worksheets using the dynamic mathematics software GeoGebra were designed reflecting various design criteria and principles (e.g., Hohenwarter & Preiner, 2008; Mayer, 2009).

Due to the prior knowledge of selected students (experiences mainly with distance-time diagrams, direct and inverse proportionality including their graphical representations but none with the explicit function concept), these worksheets primarily address the representational transfer between iconic situational models and graphical representations, a transfer that is particularly problematic (see section 2.2). This corresponds also to an idea of Leuders and Prediger (2005) that algebraic representations (equations of functions) should not be introduced too quickly. The main idea is to provide learning experiences in an early phase of learning functions possibly preventing the development of intuitive conceptions to misconceptions.

Dynamically linked, interactive representations have the potential to emphasize connections between both representations as well as functional aspects – especially the co-variational aspect. An automatic translation takes place so that the learners can concentrate on the complementary information provided by the individual representations. The iconic situational representation serves mainly as constraining representation for the less familiar graphical representation so that students may learn how to interpret graphs.

3.2 Examples

In sum, I designed ten interactive worksheets for the research study: seven applets addressing graph-as-picture errors, two including geometrical tasks about enlargement of a square, and two materials contain visualizations of motion problems.

Those part of the research project dealing with graph-as-picture errors proved to be the most varied. Therefore, I present the following two interactive worksheets, which are based on tasks from Schlöglhofer (2000). They serve best for exemplifying the various design considerations of the applied materials.

3.2.1 Billiard

The interactive worksheet "Billiard" (see Figure 2) contains an iconic representation of the situation and a GeoGebra graphics window with the option to display the graphical representation in trace mode.

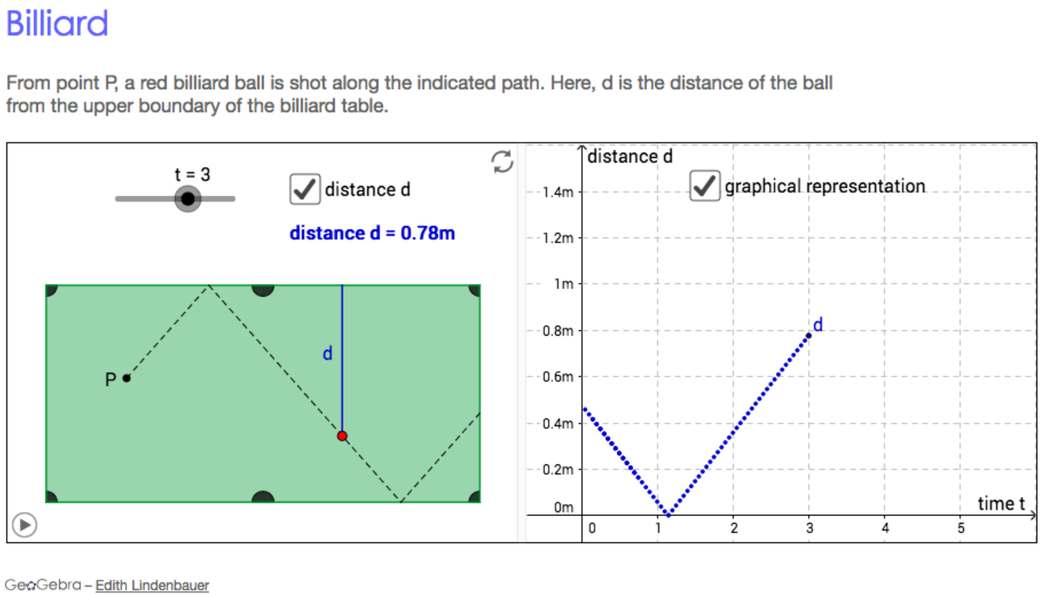

Fig. 2: Interactive worksheet "Billiard", https://ggbm.at/z2avsGqt

(Source: Adapted from "Investigating students' use of dynamic materials addressing conceptions related to functional thinking", by E. Lindenbauer, 2019, p. 2877).

The applet simulates the following situation: From point *P* a red billiard ball is shot along the indicated dashed path. The distance *d* of the ball from the upper boundary of the billiard table is a function of time *t*. As the motion of the ball is animated with a constant speed neglecting friction for simplification reasons, distance *d* forms a piecewise linear function.

When starting the animation, the ball rolls with constant speed first to the upper boundary with decreasing distance, then down to the lower boundary with increasing distance, and up again with decreasing distance. Students have the opportunity to watch initially only the movement of the ball in the left window and to make assumptions about how the distance *d* is changing qualitatively in the course of time. In this process, students have the option to pause and continue the animation or to use the slider representing time *t* to control the animation. Additionally, the situational representation provides a check box "distance *d*" for displaying corresponding numerical values of distance *d(t)*, which aims at supporting students to control their qualitative assumptions about the changing distance and to form conjectures about the shape of the graph.

On the right side of the applet, distance *d(t)* can be displayed as trace in a Cartesian coordinate system after activating the check box "graphical representation". Students are able to explore the relationship between the dynamically linked representations and to examine possible conjectures about the graph. The relational aspect of this functional dependency should be emphasized through pointwise appearance of the graphical representation in trace mode and the option to display the numerical values of time *t* and distance *d*. The dynamic feature of the animation and the linked representations should address the co-variational aspect.

3.2.2 Triangle dynagraph

The GeoGebra worksheet displayed in Figure 3 is based on a task of Schlöglhofer (2000) mentioned in his paper about the graph-as-picture misconception. Hoffkamp (2011) examined the representational transfer to Cartesian coordinates in a similar learning environment in more detail with regard to (pre)conceptions in calculus in upper secondary school level. The worksheet consists of a situational model as well as a dynagraph representation positioned underneath the situational model. During the research project, I applied a similar material displaying the same problem but utilizing Cartesian coordinates instead.

In the situational model, a triangle is displayed. The area *F* of the shaded figure left of the dashed line inside the triangle is treated as a function of *x*, which is the horizontal distance between the vertex *A* and the dashed line. Students can move the line and change the area of the colored figure. The function *F* is a piecewise-defined, monotonically increasing function.

Students have to interpret the dynagraph representation for determining approximately the numerical value of the area. After activating the check box "graphical representation", on the lower axis the respective value of distance x_0 and on the upper axis the corresponding area $F(x_0)$ are displayed. This feature provides an "input-output" perspective, which stresses the relational aspect. Advantageously for avoiding a graph-as-picture error, students cannot perceive the dynagraph representation as static "image". The option to display the trace of the function value enables students to examine the image of the function, its minimum and maximum values, and possibly also emphasizes the monotonic properties of this function.

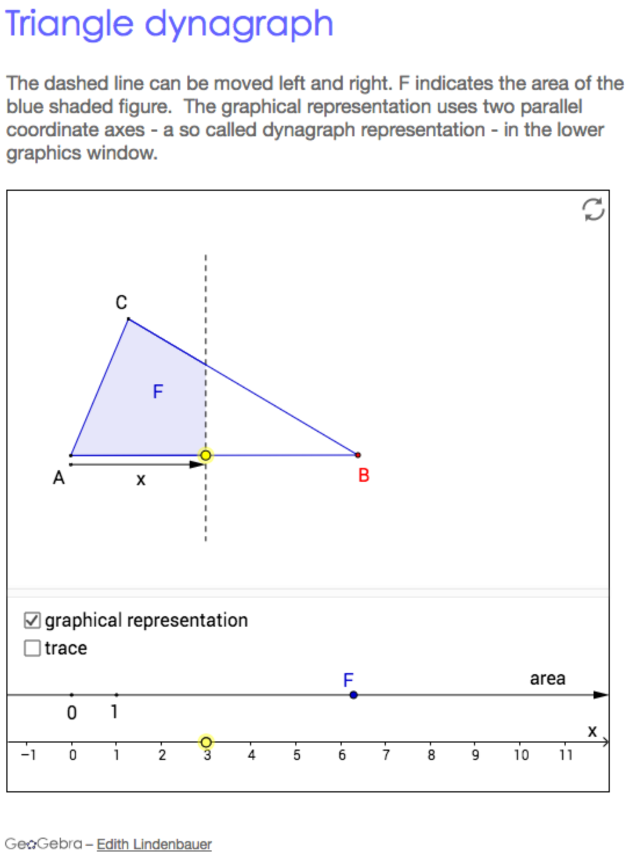

Fig. 3: Interactive worksheet "Triangle dynagraph", https://ggbm.at/wrVPlwHE

Students have the opportunity to manipulate the situation by dragging the yellow point either in the situational or dynagraph representation, thus changing the distance *x* and the corresponding function value *F(x)* accordingly. This dragging option provided in both representations and the possibility to actually vary the distance in the dynagraph emphasizes the co-variational aspect of the functional dependency. Furthermore, this representation allows students to examine the dependency of the area *F(x)* from the argument *x* because the point representing the function value on the upper axis cannot be dragged.

In sum, I designed ten interactive worksheets assembled within six topics for the research study, which will be described in the next section.

4. Research study

4.1 Research methodology

For addressing the research questions, I selected a qualitative, inductive approach. The methodological considerations led to a theory-building case study research that integrates features of Grounded Theory in a case study design, in which the students represent different cases.

4.1.1 Inductive approach

In this research project, I wanted to examine what happens when students work with interactive worksheets in order to understand whether and in which ways these materials could support students and influence students' conceptions. There already exists a variety of literature about functions, conceptions and problems in this field, and about the use of technology. However, literature review reveals that we presently do not have sufficient knowledge about students' conceptions in an early phase of learning functions (especially in the German-speaking literature) and about the basic conditions of any possible influence of these technology-based materials on the conceptions of such students. In addition, I could not identify similar research about influence of interactive representations on students' conceptions in this particular field. As there does not already exist a theory about these specific phenomena, this research project aims at inductively developing such a local theory that, for example, afterwards could guide the researcher through a follow-up design-based research project for improving the applied dynamic materials.

For developing a local theory, an inductive research approach, in which a theory is built up from data, seems appropriate (Charmaz, 2014; Teppo, 2015). In particular, I followed Eisenhardt's (1989) approach utilizing case study research for theory-building that integrates methods and features of Grounded Theory research. Instead of letting this research be guided by theory-driven conjectures, I wanted to keep an open mind about what might appear within the data in order to first develop a basic understanding about students who utilize interactive worksheets in this particular mathematical field.

Concerning Grounded Theory, I tend to follow Charmaz' (2014) stance; she states that "literature review gives you an opportunity to set the stage for what you do in subsequent sections or chapters" (p. 308). According to her suggestions, the literature review serves to reveal gaps in existing knowledge, to position the study, to argue the framing of this study, to clarify its contribution, and to be able to relate the results with literature. Literature review should lead to a critical stance towards already existing knowledge in Grounded Theory related research (Charmaz, 2014).

4.1.2 Theory building from cases

Darke, Shanks, and Broadbent (1998) mention a number of difficulties in case study research, for example, the analysis of a considerable amount of qualitative data with no standard analytical approach. One way to solve this problem is analyzing the data in a more rigorous and structured way by applying Grounded Theory analysis methods (Darke et al., 1998; Halaweh, 2012). Different researchers have already combined Grounded Theory with case study research (Halaweh, 2012; Lawrence & Tar, 2013). Concerning building a theory from case study research, I followed mainly the approach of Eisenhardt (1989), who describes the process of building a theory from case study research integrating typical features of Grounded Theory.

Eisenhardt (1989, pp. 533–545) developed a roadmap including several steps for theory-building from case studies. First, the researcher has to clear the research focus and to define the research question(s). For choosing the participants, the researcher applies theoretical sampling, in which the sample should represent a broad range of the interesting population. Afterwards, usually different data collection methods are chosen and prepared. Next, the researcher enters the field; data collection and analysis are typically intertwined and form a flexible design. Furthermore, field notes and memos are usually included in the data. For analyzing data, the author suggests first a within-case analysis to become familiar with each case and then to compare cases for searching cross-case patterns. Another strategy may be analyzing data by data source. Ideas, relationships, and hypotheses emerge from data analysis. The emerging hypotheses are constantly compared with data (and also literature) until the researcher can build a theory fitting to the data. The central idea is Grounded Theory's "constant comparison". Finally, the process stops until theoretical saturation. For pragmatic reasons (e.g., time), it is even possible to plan the number of cases in advance (Eisenhardt, 1989).

Guided by this roadmap, I developed some specific alterations due to my research purpose concerning theoretical sampling and data collection in a flexible research design. First, the research questions are strongly connected to each other and I had to examine students' conceptions before and after the use of interactive worksheets in class. Therefore, I was considering the influence of time on observational data; these considerations are related to the validity in qualitative design. According to Cohen, Manion, and Morrison (2011), "history" and "maturation" may be threats to the internal validity of this study that I had to consider. Both threats refer to an influence on the observational results beyond any intervention only because time passes during the research process (e.g., influence through regular teaching lessons during the research process, the change of students' conceptions because they are getting older) (Creswell, 2014). Due to this time-based influence on the development of students' conceptions, I had to guarantee that the whole data collection process did not last longer than two months.

Second, I changed the daily routine in students' school due to an intervention in school life when using the interactive worksheets, doing tests and conducting interviews. For organizational and ethical considerations, it was sensible to include a whole class instead of single students from different schools. Therefore, I designed a fixed instead a flexible research study. For ensuring theoretical sampling and theoretical saturation as important features of a theory-building approach, I selected two classes from a new secondary school usually attended by a wide range of low- and high-achieving students. Furthermore, I collected as much data as possible, which served as data pool for constant comparison during the data analysis. According to Vollstedt (2015), this is a reasonable way to guarantee theoretical saturation if the data collection process does not include multiple cycles.

4.2 Data collection methods

Several types of data were collected in this study by diagnostic tests, diagnostic interviews, students' worksheets on paper, and observations during the intervention.

The first research question enquires about students' conceptions concerning functional thinking. For observing these conceptions, I utilized a diagnostic test to obtain an overview of all participants, and afterwards I conducted several diagnostic interviews for an in-depth investigation. The second research question seeks to examine the influence of interactive worksheets on students' conceptions. I selected the following data sources to approach this question: (i) a recorded observation (audio, video, and screen recordings) of students while they are working with the dynamic materials together with written comments and answers on paper worksheets attached to the interactive worksheets, (ii) a diagnostic test after the observation, and (iii) diagnostic interviews with students based on the observation and test results.

4.2.1 Diagnostic tests

According to Gurel et al. (2015), diagnostic tests are commonly used to diagnose students' (mis-)conceptions. The paper-and-pencil diagnostic tests designed for this project are based on different tasks from literature concerning conceptions (De Bock, Van Dooren, Janssens, & Verschaffel, 2002; Schlöglhofer, 2000) as well as a test instrument called CODI (Nitsch, 2015). Furthermore, each task includes open-ended questions that aim at revealing students thinking processes, for example, students should explain their solutions or describe and reason their considerations.

Diagnostic test 1 approaches the first research question about students' conceptions concerning functional thinking, and *diagnostic test 2* addresses the second research question about a possible influence of the dynamic materials on students' conceptions. It resembles the first diagnostic test using slightly modified tasks. Both tests aim at getting an overview of students' erroneous and correct answers related to main problems in the area of functional thinking that may refer to their conceptions.

The first diagnostic test contains 10 tasks combined within three topics: graph-as-picture error, distance-time diagrams addressing the slope-height confusion, and illusion of linearity in tasks related to the enlargement of two-dimensional figures. During the research process, I decided to focus on the representational transfer between verbal description and/or iconic situational model and graphical representation. Therefore, seven examples within the first two topics address this kind of representational transfer. As participants' prior knowledge did not include linear functions, I included three examples addressing illusion of linearity in tasks about enlargement of areas or two-dimensional figures.

4.2.2 Diagnostic interviews

As mentioned in the previous section, I decided to conduct diagnostic interviews to approach – together with diagnostic tests – the two research questions. Within the data collection process, diagnostic interviews were conducted twice after both diagnostic tests (see Figure 4). The first interviews were related to diagnostic test 1 and aimed at addressing the first research question about students' conceptions; the interviews after diagnostic test 2 should, together with students' test results, support the examination of the second research question about any influence of working with dynamic materials on students' conceptions. While diagnostic tests should provide an overview of all participants' conceptions regarding to the selected tasks, diagnostic interviews were held with students depending on their test responses to obtain an in-depth view of their individual conceptions.

Diagnostic interviews were semi-structured. Based on the interviewee, each interview guide included fixed questions or tasks to be examined. The actual course of the interview, however, depended on participants' answers, allowing additional and clarifying questions as well as variations of already planned questions. This kind of interview aims at establishing a dialogue between the interviewer and the student to reveal students' understanding of mathematical concepts or ideas (Hunting, 1997).

4.2.3 Observation

The observation took place during a three-lesson-intervention, when students worked in pairs with developed interactive worksheets (see section 3). While working, 10 students were audio- and videotaped and the screens of their laptops were recorded. In addition, students' paper worksheets accompanying the dynamic materials were collected. As I wanted the recording procedure to be as unobtrusive as possible for the participants, I decided to utilize those laptops the students were already working with for recording.

In this research project, I was mainly interested in the influence of these interactive worksheets on students' conceptions. Although the teacher's ability to integrate such materials in mathematics education is an important factor for effective learning (Drijvers et al., 2016), first we should understand what happens when students work with these worksheets without any teacher guidance. Therefore, my role as researcher was mainly as nonparticipant observer (Creswell, 2014), except for the organization of the intervention, as supervisor of the participating students, and contact in case of non-mathematical questions. Both mathematics teachers of the participating classes were not part of the observation process to ensure that they did not influence participants in their learning processes.

4.3 Research design

The research study consisted of different phases. *Pilot study A* was the first phase of the study aimed to evaluate the technical details of the recording procedures, to choose the tasks for the diagnostic tests and the interactive worksheets for intervention, and to pilot the interview procedures. The second *pilot study B* consisted of one complete data collection process in an eighth grade of a new secondary school; the same process took place for the *data collection* in a seventh grade of a new secondary school. After the data collection, the data was transcribed and analyzed with qualitative methods described later (Lindenbauer & Lavicza, 2017).

The above described data collection methods were arranged to a research design including five data collection stages: (1) diagnostic test 1, (2) diagnostic interviews, (3) intervention, (4) diagnostic test 2, and (5) diagnostic interviews. Figure 4 presents an overview of the data collection process and the number of participants in each stage.

First, all 28 selected students participated *diagnostic test 1*. Based on a follow-up analysis and students' mathematics grades, I divided the students into two groups. For the first group, I selected ten students combined in five pairs for participating the observation process of the planned intervention. They were not selected as interviewees for the first diagnostic interviews before the intervention, because I did not want to influence their conceptions additionally by the interviewing process before they were working with interactive worksheets. Every pair was formed homogeneously with similar mathematics grades and comparable test responses. Homogeneous pairs should guarantee that most influences on students' conceptions is due to working with interactive worksheets and not because of learning with a higher achieving colleague. Together, these students represented the range of achievement levels in mathematics from all participating students.

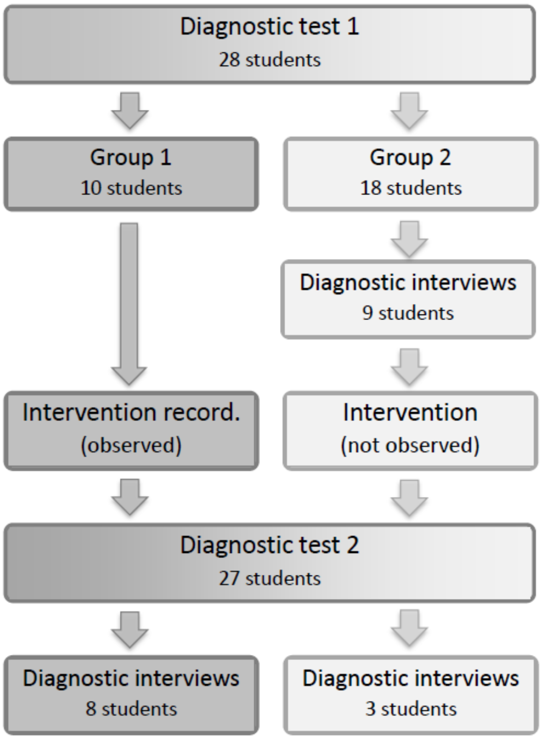

Fig. 4: Data collection process

For *diagnostic interviews*, nine students of the second group were chosen depending on their test responses so that their – incorrect – results represent a wide range of different conceptions related to the various tasks of the test. During the *three-lesson-intervention*, students worked in pairs with interactive worksheets addressing different topics guided by accompanying tasks. While working, students of group 1 were audio- and videotaped and the screens of their laptops were recorded. Also, students' paper worksheets were collected. Group 2 students worked in another school's computer lab supervised by their mathematics teachers. Both teachers were instructed before the intervention process in order to guarantee that the participants were not influenced in their learning processes by the teachers' help.

After completing the intervention, *diagnostic test 2* with slightly altered tasks was conducted. Based on the observational data and an analysis of the test results, I selected 11 students for *diagnostic interviews* to gather additional data for approaching the second research question about the influence of the dynamic materials on the students' conceptions. After the data collection process, students' names were changed to pseudonyms to preserve confidentiality and anonymity.

4.4 Participants

The study was conducted with two seventh grade classes of a rural new secondary school consisting of 28 students aged 12 or 13 within the second semester of the school year. Rural new secondary schools in Austria usually have the most diverse student population concerning achievement levels. This is especially true for the selected school in this study; thus, this choice offers the possibility to examine as many different students and their conceptions as possible and to support internal validity of this research project.

As specified by the Austrian curriculum, students in seventh grade should be acquainted with the representations of functional relationships as graph, formula, and table as well as two types of functions (direct and inversely proportional models). Therefore, they should be able to interpret Cartesian coordinates and to study graphical representations, but they are not accustomed to the concept of function. According to the mathematics teachers of the selected classes, at the time the study took place participants mainly had worked with direct and inverse proportionality focusing on real-world applications and distance-time diagrams. Students had some experience in working with tables, reading function values from graphical representations, interpreting graphical representations, and creating graphs mainly using the software GeoGebra.

4.5 Data analysis

After the data collection, I divided data by source, transcribed and analyzed it by using the analysis software MAXQDA. During data analysis, I followed coding procedures based on Grounded Theory including initial, focused, and theoretical coding guided by the idea of constant comparison. Furthermore, I intertwined these coding procedures with Eisenhardt's (1989) suggestions to analyze data within-case and cross-case.

For the *first research question* about students' conceptions, I started with a brief within-case analysis of the individual students' test responses. Then, I conducted initial coding of students' test results and explanations as well as transcribed interview data. As mentioned before, the tasks of the first diagnostic test can be grouped within the topics "graph-as-picture error", "distance-time diagrams", and "illusion of linearity". Due to differences between the tasks of each topic, which address separate students' conceptions, I analyzed the data following this structure.

During focused coding, I reorganized and structured the emerged codes, integrated codes from interview and test data, subsumed them within categories, and tried to find patterns and relationships between categories. Concerning the first topic ("graph-as-picture error"), after final theoretical coding, not core categories but diagrams representing the structure of the representational transfer emerged as condensed result of the data analysis for each task. Finally, the results concerning this topic were compared and synthesized.

Regarding the other two topics, related tasks were analyzed together due to their respective similarities and connections. Similar to the first topic, I started with reanalyzing test data based on initial codes, followed by developing categories, enriching and supplementing them by interview data, and examining structures and relationships among the emerging themes. Again, I did not identify core categories, but major themes answering the first research question resulted from data analysis.

When approaching the second research question, I divided the data by source and started with initial coding, first of intervention data including paper worksheets, then diagnostic test responses, and finally interview data (transcriptions as well as excerpts and memos). Due to the amount of data, I coded intervention recordings for each topic of interactive worksheets that students utilized separately in MAXQDA. Moreover, I created brief within-case descriptions of the participants comparing their results from the first to the second diagnostic test. If available, I additionally included interview data within these descriptions.

Focused coding contained several cycles of comparing and restructuring codes and data, developing categories, comparing categories with each other, with new emerging codes, and categories from, for example, different data sources by constant comparison. Finally, I compared and integrated emerged categories from all data sources until first ideas, patterns, and hypotheses were revealed. In the final phase of theoretical coding, I integrated and synthesized categories from focused coding as well as patterns between them. I identified the core category around which other categories were related and analyzed these relations. Emerging hypotheses, conjectures, and ideas were examined until the results appeared to be well established.

5. Key results and discussion

In this section, I highlight selected findings and present summarized key results from the research project regarding both research questions.

5.1 Students' conceptions

As the tasks of the diagnostic test can be grouped into topics, I now present summarized results within the topics named as "graph-as-picture error" and "distance-time-diagrams".

5.1.1 Topic: Graph-as-picture error

Figure 5 displays one task of the first diagnostic test based on a task from Schlöglhofer (2000). It addresses a graph-as-picture error and resembles the situation presented in the interactive worksheet "Triangle dynagraph" (see Figure 3).

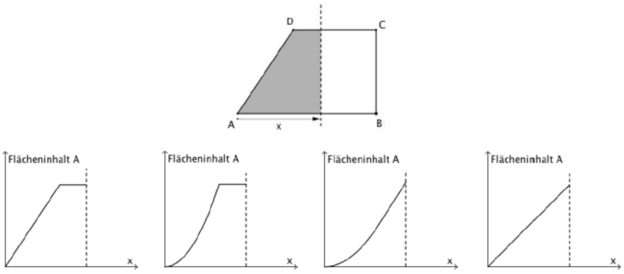

Fig. 5: Diagnostic test 1, task "Area"

This task presents a trapezoid; students should imagine drawing the dashed line from vertex *A* to the right by the distance *x*. In doing so, they should imagine how the area of the grey marked figure left of the dashed line changes. Afterwards, students had to choose one graph out of four representing the area of the grey marked figure as a function of distance *x* and to reason their solution. The third graph represents the correct solution, while the first presents a solution representing a graph-as-picture error. The second graph combines a correct graph with a graph-as-picture error, and the fourth solution visualizes a linear growth of area.

This problem emphasizes the co-variational aspect as students have to examine the changing area of the grey shaded figure when the dashed line is moved the distance *x* to the right, and both quantities have to be considered mutually. The main focus lies on the qualitative change of the area. This representational transfer was also examined in more detail by Hoffkamp (2011) with regard to (pre)conceptions in calculus, and it is considered particularly difficult. In contrast to other tasks of the diagnostic test, no real-world situation is presented but an abstract task, where students have to mentally visualize the changing situation.

Students' responses revealed various levels of conceptual understanding. Figure 6 visualizes a categorization of students' solutions and argumentations. The arrows represent the direction of the representational transfer from the situational model to the function graph, and the categories are arranged according to the correctness and elaborateness of students' understanding.

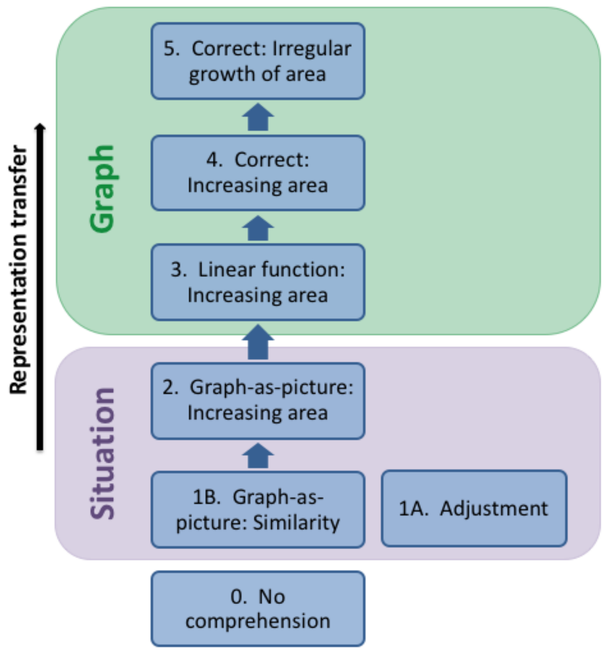

Fig. 6: Solution categories task "Area"

Six students were not able to solve this task, most of them because of a lack of comprehension. Due to this relative high number, I included a category "No comprehension", which indicates the difficulty of this task.

The next two categories 1B and 2 represent the choice of the first graph addressing a graph-as-picture error. Christoph (Category 1B) explained: "Because it has to be so, the diagram must look the same way as the grey shaded area" [2]. Either the students marked like Christoph the similarity between both representations, or they already recognized an increase of area but "remained" at the shape of a trapezoid. In the latter case, students proved able to mentally animate the situation. For example, Hannah explained as follows: "Because more and more is getting grey. However, the area/shape does not change. If we draw the line to *B*, everything will be grey, but it is still the same shape". Similar to the other students on this level, Hannah was able to mentally animate the situation and to determine that the area of the grey shaded figure is increasing when moving the dotted line accordingly. However, these students confused the area of the figure with the shape of the trapezoid and thus were not able to abstract the dependency of the area from the distance *x*, which led to a graph-as-picture error. Summarized, these answers revealed reasoning from a situational perspective of students, who did not manage to transfer the situational model into a function graph. Also answers assigned to "1A. Adjustment" were based on a situational perspective, where students tried to find similarities between both iconic representations or tried to adjust (parts of) the iconic situational model to the chosen graph. One characteristic student answer was given by Amelie: "If you draw the dashed line from vertex *A* to *x*, it is a straight line." Interview data further revealed her focus on the arrow labeled *x* (see Figure 5). As this is a straight line, she chose the linear function out of the four possible solution choices due to its visual similarity to this arrow. It could be discussed if her answer represents a kind of graph-as-picture error.

The third category includes answers of students who selected the linear function and described qualitatively correct that the area increases when you draw the dashed line to the right but did not recognize the irregular change of the function value. For example, Barbara explained: "Because when you continue to draw this line, the side *x* and the area gets bigger and bigger". Categories 4 and 5 consist of solutions of students who correctly chose the third graph with differently elaborate descriptions of the increasing area when mentally animating the situation. For instance, Harald described the irregular changing area: "First, it [the area] gets slightly larger, then stronger, and finally the increase remains constant."

Data analysis revealed that students with solutions of categories three to five achieved transfer to a graphical representation by recognizing an increasing function value, and these solutions were essentially correct, whereas the other students reasoned from a situational perspective. This gap visualized in Figure 6 before the last three categories indicates a "comprehension gap" that we can assume between students who either achieved to translate the functional dependency to a graphical representation and students who did not. As similarly outlined by Nitsch (2015, pp. 282–291), this comprehension gap appears to be an obstacle that students have to overcome to be able to reason from a graphical perspective.

In addition to the task "Area", the topic "graph-as-picture error" of the research project included two further tasks ("Skier", "Billiard") also based on examples from Schlöglhofer (2000). For each of these tasks, several levels of conceptual understanding emerged during data analysis that represent the translation process from situational to graphical representations. Furthermore, these levels indicate that students reasoned from different perspectives (situational, graphical). There appears to be a relation between students with incorrect solutions and their tendency to explain from a situational perspective and to prefer a pointwise, static view on functional dependencies. Furthermore, also results from the other tasks "Skier" and "Billiard" indicate a comprehension gap between situational and graphical representations and not a continuous transfer. Possible reasons for this gap could be students' difficulties in understanding and interpreting Cartesian coordinates or students' inability to focus on more than one feature or variable. The latter can be examined under the more general term of covariational reasoning as described by Johnson, McClintock, and Hornbein (2017).

The joint analysis of the above-mentioned three tasks regarding the graph-as-picture error in the research project led to following conclusions: For being able to successfully translate from verbal and/or iconic situational to graphical representations, it appears that students should manage the following steps: (i) to understand the presented situation, (ii) to visualize the presented situation, (iii) to identify the dependent variable, (iv) to describe (verbally) the behavior of the dependent variable as a function from the independent variable in a qualitative way, (v) to translate the basic course of the functional dependency to a graphical representation (e.g., qualitative correct change of the function value), and (vi) to represent changes of the dependent variable in a correct way (e.g., qualitative correct change of slope). The above-mentioned comprehension gap would thus occur between steps (iv) and (v) (Lindenbauer, 2018).

In essence, several students' problems, conceptions, and solution strategies appeared during data analysis, for example: problems in understanding the presented situation, especially when students were not able to rely on everyday experiences; misinterpretations of the presented situation or the dependent variable; various forms of graph-as-picture errors probably caused by a confounding influence of the iconic situational model; illusion of linearity; and influences of prior knowledge. Moreover, students' tendency to search for visual or structural similarities between both representations was frequently observed, in particular when students struggled with problem-solving.

5.1.2 Topic: Distance-time diagrams

Regarding the topic "distance-time diagrams", the majority of students was able to solve these problems correctly, possibly due to their prior teaching experiences in this field. However, several student errors, problems, solving strategies, and influencing factors on their learning processes emerged from data analysis. These themes contain (i) various forms of slope-height confusions, (ii) language-based misinterpretations, (iii) graph-as-picture errors, (iv) errors due to inaccurateness when determining function values from graphs, (v) problems of missing scaling, (vi) misinterpretations of axes, (vii) tendency to ignore information, again (viii) search for visual and/or structural similarities, and (ix) an influence of real-world experiences. Various reasons could cause these student problems and errors; for example, an influence of informal language (e.g., when students interpret "highest speed" as "leading" or "winning") may lead to a slope-height confusion.

The confusion of slope and height indicates that students focus on the position of the function graph instead of the slope at the respective point of time or during a period of time. For example, Jasmin appeared to underlie a specific kind of slope-height misconception. Figure 7 presents task "Runner" of the first diagnostic test, in which students had to identify which runner is the fastest in the period from 4 to 5 seconds. By choosing solution 1, Jasmin made a slope-height error but only in this example. She solved correctly two further corresponding single-choice tasks providing, among others, the following explanation: "Because at 5 [seconds] … the graph is highest".

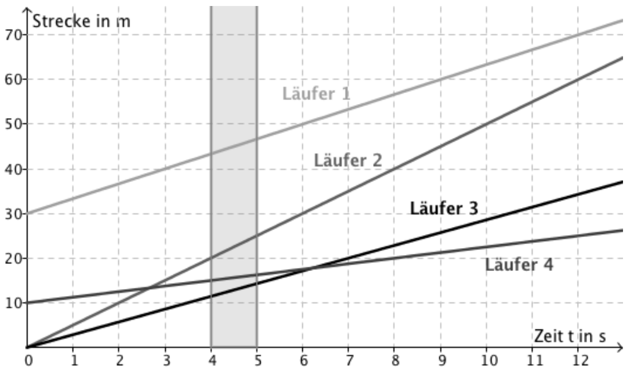

Fig. 7: Diagnostic test 1, Task "Runner"

(Source: Adapted from "Diagnose von Lernschwierigkeiten im Bereich funktionaler Zusammenhänge", by R. Nitsch, 2015, Wiesbaden: Springer Spektrum.)

At first, I thought it to be a correct perception of the highest slope. However, her solutions did not reveal a pattern described, for example, by Nitsch (2015) with respect to slope-height confusions, but they had one feature in common: In all three tasks, Jasmin chose the graph with the highest function value at the end of the observation period. A subsequent interview revealed, that she actually concentrated on the function value at the end of the observation period, as it were, on the winner of a race. In effect, Jasmin was able to solve tasks correctly based on an incorrect conception because in two of the three utilized tasks, the fastest object had also the highest function value at the end of the observation period

This example together with further student explanations in the first diagnostic test showed that faulty activated conceptions led to correct answers in single choice items; only student explanations revealed incorrect thinking processes. Results thus made visible that standardized single-choice test items were not always able to detect a corresponding incorrect conception.

5.2 Influences of dynamic materials

Various categories emerged from the common analysis of observation data from intervention as well as data from the second diagnostic tests and interviews that can be subsumed under several topics, for example, influencing learning process, misinterpretations and sources of difficulties, dynamic materials and representations, and potentials and problems of utilizing dynamic materials. In brief, there appears to exist a supporting influence of working with interactive worksheets on students' conceptions; however, the extent of this influence seems to depend on students' intuitive conceptions and achievement levels. Following, I highlight some summarized main results from the project.

5.2.1 Influencing learning process

As can be seen in Figure 8, when students utilized dynamic materials for solving tasks, they observed what happened within the dynamically linked representations either more actively by *manipulating* dynamic representations (e.g., experimenting by moving sliders) or more passively when *observing* an animated movement. The number and variety of codes assigned to recorded data suggest an influence of *achievement level* on how diversely students worked with these dynamic materials; in particular, higher achieving students seemed to utilize these materials in more varied ways. For instance, Wolfgang and Harald, the highest achieving students who worked together during the intervention, used the trace mode to mark extrema of the function within a dynagraph representation (Interactive worksheet "Billiard dynagraph", https://ggbm.at/hSP3H9wv), something that no other student did. In contrast, lower achieving students mainly utilized this material passively (e.g., by watching animations). Possibly, this student behavior could be connected to students' use of visual representations in problem-solving processes as described by Stylianou and Silver (2004). These researchers compared novice and expert students' behavior and realized that the latter utilize visual representations in a wider variety of problems and in more varied ways (Lindenbauer, 2019).

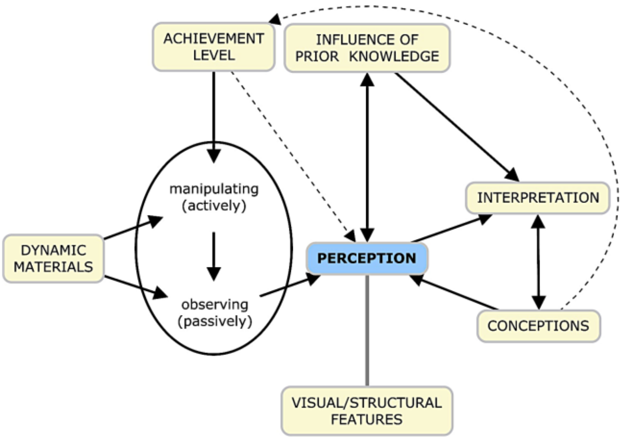

Fig. 8: Influencing factors of learning process with dynamic materials

The way the students utilized the dynamic materials influenced what they observed and consequently what they perceived. As visualized in Figure 8, students' learning processes include several factors. The data analyses reveal perception as a main category of working with interactive worksheets. In this context, perception means that students consciously observe something and take it further into account for considerations or interpretations. In particular, students appear to focus on *visual or structural features* of iconic situational and graphical representations. This student behavior becomes especially apparent when students have difficulties in solving tasks. Therefore, such features (e.g., shape of graph, trace) seem to influence considerably students' perception and thus their learning processes.

Repeatedly, there was a difference between what students were supposed to observe and what they really perceived, especially average and low achieving students. The dashed arrow in Figure 8 visualizes a possible existing influence of students' achievement levels on their perceptions. For example, Konstantin and Mario worked with the worksheet "Triangle", presenting the same situational model as in Figure 8, for examining the question how the area of the shaded figure changes when moving the yellow point from right to left.[2]

> Konstantin: (While moving the dashed line from left to right and watching the increasing area): So, it [the area] is small. Then it gets big, big, big, and then we have the entire big one.
>
> Mario: So, it is dragged into the length.

Konstantin immediately focused on the area of the shaded figure, while Mario's remark could be interpreted that he perceived a change of form and thus focused on the visual feature "shape" and not the intended variable.

In essence, combined data analysis regarding the second research question of the overall project indicates an intention-reality discrepancy between the mathematical content the dynamic materials are intended to visualize and what students really perceive, especially when students focus on visual features of interactive worksheets. One reason for this discrepancy could be that students would need more profound knowledge and stable conceptions about Cartesian coordinates and their interpretation as well as about representational transfers.

The interpretations as well as perceptions of the dynamic materials are affected by students' *prior knowledge and experiences*. Observational data repeatedly demonstrated that students tried to link perceived features to their existing knowledge (e.g., geometric figures, distance-time graphs, proportionality) or everyday experience. For instance, the following conversation took place when Carina and Sarah first encountered the graphical representation in dynamic worksheet "Triangle" (see upper graphics window in Figure 3). They should examine the question of how the shaded area changes when moving the dashed line from left to right.

> Carina: (moves the dashed line from vertex *A* to vertex *C*); a triangle. So, if it [dashed line] is at vertex *C*, if it is like that …, then it is a triangle. If you make it so (moves the dashed line to vertex *B*), then it is also a triangle. And so (moves the line backwards in between *B* and *C*) it is, um …
>
> Sarah: Here, it is a polygon.
>
> Carina: A quadrilateral.

Especially Carina first focused on the form of the shaded figure instead of its area, and she compared the perceived forms with already known geometric figures. Further data also analysis indicated, that especially when students had problems in understanding and solving the presented tasks, they tended to rely on their prior knowledge for problem-solving.

Literature also discusses this relation of prior knowledge with students' perceptions and conceptions. According to Roschelle (1995), who examined learning in interactive environments, prior knowledge actually influences students' perceptions and conceptions such as Carina's focus on the form of the figure instead of its area in the above-mentioned example. Their individual prior knowledge and experiences affect how students perceive visual representations and thus how they interpret them (e.g., Cook, 2006; Roschelle, 1995). Especially perceptual set theory of cognitive psychology discusses such individual influence on students' perception when selecting and interpreting information (McLeod, 2010).

5.2.2 Misinterpretations and sources of difficulties

Several sources of *difficulties* and *misinterpretations* of students when working with dynamic materials and related tasks could be identified in this project. The outlined difficulties include language-related problems, difficulties during various steps of the representational transfer, incorrect observations, misinterpretations of perceived features, and confusions of, for example, variables or axes. Further misinterpretations can be connected to misconceptions of students, such as graph-as-picture errors, illusion of linearity, and slope-height confusions. Moreover, results indicate an influence of students' individual prototype of a function as well as incorrect physical conceptions.

Two further misinterpretations, which I call "adjustment" and "reflection", are strongly related to students' tendency to look for visual or structural similarities. *Adjustment* refers to students' behavior of trying to adjust two representations of a functional dependency (e.g., verbal description and graph) incorrectly by making them fit in some way. It is especially apparent in single-choice item tasks and could be based on a distracting influence of incorrect solutions or a lacking conceptualization of Cartesian coordinates.

Reflection as misinterpretation evolved during this research project and is probably related to a graph-as-picture misconception. Figure 9 depicts the task "Billiard" from the second diagnostic test. In this task, a billiard ball moves along the indicated path, and students should consider the distance of the ball from the upper boundary of the table as a function of time.

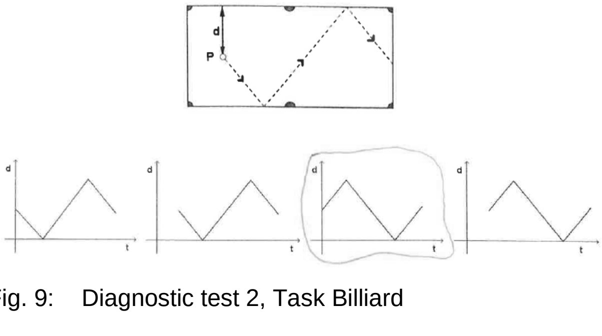

Fig. 9: Diagnostic test 2, Task Billiard

Pia chose the correct third solution and reasoned: "I have thought it has to be reversed". Although she chose the correct solution, her test response was not meaningful enough to determine her conceptions, and I started an interview by referring to it.

Interviewer: Why reversed?

Pia: Because the other solution is not there, the right one.

Interviewer: Which right one?

Pia: The upper one (*moves along the path of the billiard ball in the situational model*). And then I thought it has to be reversed, mirrored.

The ensuing dialogue revealed that by "the right one", Pia meant a graph resembling the iconic model of the billiard table. She was looking for the graph representing a graph-as-picture error and overlooked this particular solution in Figure 9. Her explanations imply a graph-as-picture misconception and also reveal a specific solution strategy. Pia did not consider the change of the dependent variable in course of time but perceived a specific visual feature – the form of the path in the iconic situation model. Afterwards she looked for a solution globally corresponding to the situation. As she overlooked that particular solution, she tried a further idea and chose the graph reflecting the iconic situational model, something she experienced when working with the interactive worksheets. In fact, both ideas resemble each other as one global feature of the iconic model – its form – is interpreted as potential graphical representation. In sum, data indicate that Pia interpreted a specific perceived feature influenced by a "graph-resembles-picture" conception.

This kind of misinterpretation is most likely induced by the specific design of interactive worksheets the students worked with in which the iconic situational models and function graphs appear to mirror each other (e.g., interactive worksheet "Billiard" presented in Figure 2). As a result, incorrect interpreted visual or structural features could induce new misconceptions.

5.2.3 Potential and problems of dynamic materials

Dynagraph representations emphasized certain characteristic properties of functions such as fixed or inflection points, which several students observed without being explicitly questioned about. For example, Franziska implicitly recognized the inflection point of the functional dependency presented in interactive worksheet "Triangle dynagraph" (see Figure 3) when asked to describe how the function value changes when x increases. She answered:

> Franziska: Here it [area] is increasing faster (*moves the x value from x = 0 to x = 1*) . . . and then slower again.
>
> Interviewer: Do you have any idea from where it slows down?
>
> Franziska: So, from here on (*moves dashed line to vertex C*).

As the dialogue reveals, Franziska identified the inflection point from which on the change of rate decreases. The "speed" of the point representing the function value compared with those of the point representing the argument *x* seemed to be a feature easily perceived. Therefore, this dynamic characteristic of dynagraphs could be utilized in mathematics teaching to assist students in accessing graphical representations. Possibly, dynagraphs could support students as step between translating directly from situational model to representations applying Cartesian coordinates (Lindenbauer, 2019).

Results concerning the use of dynagraph representations reveal that they appear to be intuitive for students to interpret and thus provide an easy access to graphical representations. As outlined in literature, they actually seem to emphasize the co-variational aspect of functional dependencies. During the research study, a small number of minor problems regarding how to read function values appeared and thus should be considered when applying dynagraph representations in school. For these reasons, characteristic features of dynagraphs could be utilized in mathematics teaching to support students in accessing graphical representations, possibly in scaffolding the representational transfer to Cartesian coordinates.

Several potentials and problems of working with dynamic materials emerged from data analysis. *The visualizing function* of these materials can support students in understanding and visualizing a situation as well as for identifying and describing the dependent variable. Depending on students' prior conceptions and their understanding of Cartesian coordinates, the dynamic materials utilized in this project seem to have an *adaptational influence* on students' conceptions. In other words, they appear to support students in improving descriptions of functional dependencies (e.g., leading to qualitative correct ideas about changing slopes of function graphs). Results related to the first research question revealed a comprehension gap as obstacle for students to overcome during the representational transfer (see section 5.1). Data analyses indicate that if students are basically not able to understand and interpret graphical representations in Cartesian coordinates, they could rather not overcome this comprehension gap by working with interactive worksheets without teacher guidance. Apparently, they would profit from teachers' assistance to help them reflect and reconsider their perceptions and interpretations. For students who are basically able to make representational transfers, dynamic materials seem to be helpful because they appear to induce adaptations of students' conceptions.

However, these materials could also *induce new misconceptions* especially when only superficially perceived and incorrectly interpreted. Incorrect interpreted visual or structural features could influence students' conceptions in a negative way and lead to such new erroneous conceptions. Therefore, data analysis results reveal a potential danger of working with interactive worksheets without teacher guidance, especially for lower achieving students who are not able to perceive mathematically relevant features and to interpret them basically in a correct way.

6. Conclusion

Different levels of students' conceptual understanding emerged during analysis of the first diagnostic test and interview data. These levels represent the translation process from situational model to function graph and include a comprehension gap for students between both representations. In school, teachers should be aware of this gap as an obstacle that students have to overcome. For successfully performing such translation processes, students should be able to understand Cartesian coordinates and interpret such abstract representations of a functional dependency. Furthermore, student explanations made visible, that standardized single-choice item tasks were not always suitable for detecting a corresponding incorrect conception, especially for students with little experience concerning functions.

Results seem to reveal that the extent of influence of these materials on students' conceptions depends on the intuitive conceptions of students and their achievement level. When utilizing dynamic materials, teachers should be conscious of students' tendency to perceive and interpret visual or structural features of both discussed representations even when they are not relevant and not always in a correct way. As stated in Lindenbauer and Lavicza (2017), the dynamic materials seem to be more appropriate for higher achieving students when working without teacher guidance, whereas lower achieving students might profit of teachers' assistance to reflect their perceptions and interpretations. A further question to consider when teaching with interactive materials is how to draw students' attention to relevant features for mathematical learning and to support them in reflecting their ideas. In addition, I would advise utilizing dynagraphs in mathematics teaching because they seem to provide an easy and intuitive access to representations. Possibly, they could also be used to scaffold the interpretation of graphs in Cartesian coordinates.

How to improve the interactive materials and especially how to implement them in regular teaching are additional questions that emerged during this research project. Design experiments and instrumental approach respectively could be suitable frameworks to investigate these questions further in other studies.

Remarks

[1] Translation of the author

[2] All students' written and spoken comments are originally in German and translated by the author.

Thanks

I would like to thank the reviewers for their valuable and constructive remarks and comments.

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Address of authors

Edith Lindenbauer
PH Oberösterreich
Fachbereich Mathematik
Kaplanhofstraße 40
A-4020 Linz
edith.lindenbauer@ph-ooe.at

Appendix