"Is mathematical knowledge certain? – Are you sure?"

An interview study to investigate epistemic beliefs

von

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Abstract: The goal of the study presented in this article is to identify epistemic beliefs about mathematics as a scientific discipline with a focus on *certainty of knowledge*. The topic is discussed from both a psychological point of view and from the philosophy of mathematics. An interview study was conducted to investigate on the question whether domain-general conceptualizations of the certainty dimension are applicable to the domain mathematics. The recorded positions and arguments ask for a domain-specific conceptualization of certainty.

Kurzfassung: Das Ziel der in diesem Artikel präsentierten Studie ist die Identifikation von epistemologischen Überzeugungen über die Mathematik als Wissenschaft mit einem Fokus auf *Sicherheit von Wissen*. Das Thema wird sowohl aus psychologischer Sicht als auch aus Sicht der Mathematikphilosophie diskutiert. Mittels einer Interviewstudie wurde untersucht, inwiefern sich generelle, domänen-übergreifende Konzeptualisierungen der Sicherheits-Dimension auf die Domäne Mathematik übertragen lassen. Deren Ergebnisse weisen auf eine Domänen-spezifische Konzeptualisierung der Sicherheit von Wissen hin.

1 Introduction

To understand the conditions, the processes and the outcome of tertiary education it is necessary to capture the knowledge and beliefs of students by developing theories and instruments that describe relevant competence areas (Blömeke, Zlatkin-Troitschanskaia, Kuhn, & Fege, 2013). For teacher education, several instruments for assessing competencies on a large scale have been developed in the past decade (Blömeke, Suhl, & Döhrmann, 2013; Baumert et al., 2010; Hill, Rowan, & Ball, 2005). Despite these developments, many aspects of teachers' competencies still lead to a controversial discussion. In particular, there are many open questions as to the structure of the beliefs of future mathematics teachers (Pajares, 1992; Grigutsch, Raatz, & Törner, 1998; Bernack, Leuders, Holzäpfel, & Renkl, submitted).

The principal goals of the project reported in this article (which is embedded within the research initiative "Modeling and Measuring Competencies in Higher Education", cf. Blömeke, Zlatkin-Troitschanskaia, et al., 2013) are to identify epistemological beliefs about mathematics as a scientific discipline; and subsequently to develop the instruments to do so reliably and economically. In this paper, we focus on a first step of this process: A preliminary interview study was conducted to identify epistemic beliefs of people dealing with mathematics (e.g., teacher students or mathematicians) in detail. Even though there exists a multitude of questionnaires and other instruments to measure epistemic beliefs, we chose interviews for reasons of validity. Our research will be exemplified by the topic of *certainty of knowledge* which is central to mathematics (Kline, 1980).

2 The structure of epistemic beliefs

Epistemology is a branch of philosophy that deals with the nature of human knowledge, including its limits, justifications, and sources (cf. Arner, 1972, ch. I). Research on personal epistemology as a branch of both psychology and education research examines the development of epistemic beliefs (which are beliefs about the nature of knowledge and knowing) and the way these beliefs filter perceptions, influence learning processes, and direct actions.

In mathematics education, research on beliefs rarely uses the terms "epistemology" or "epistemic" (e.g., these are only briefly mentioned by Thompson, 1992, or Philipp, 2007). Instead, research on this topic is sometimes assessed under the construct of "mathematical world views" (cf. Schoenfeld, 1992; Grigutsch, Raatz, & Törner, 1998).

Nature of Knowledge (what one believes knowledge is)		***Nature or Process of Knowing*** (how one comes to know)	
Certainty of Knowledge	***Simplicity of Knowledge***	***Source of Knowledge***	***Justification of Knowledge***
Assumptions about the certainty or tentativeness of knowledge.	Assumptions whether knowledge consists of simple facts vs. it is a complex network of information.	Assumptions whether knowledge originates outside the self and resides in external authority vs. assumptions of the self as knower, with the ability to construct knowledge.	Assumptions about how to evaluate knowledge claims, the role of empirical evidence, experts, and how to justify knowledge claims.

Table 1: Areas and dimensions of epistemological beliefs (cf. Hofer & Pintrich, 1997)

Authors	Methodology
Buelens, Clement, & Clarebout (2002)	They used a thirty-item questionnaire to study the correspondence of conceptions of knowledge and learning with conceptions of instruction.
Hofer (2004a)	She used think-aloud protocols and retrospective interviewing to examine how students engage in epistemic metacognitive processes during online searching for simulated class assignments.
Hofer (2004b)	She used field notes from observations and interviewed students at the beginning and the end of this observation that lasted one semester.
Hofer & Pintrich (1997)	They conducted a meta study and summarized the results of other studies.
Kardash & Howell (2000)	They used a 42-item version of Schommer's (1990) "Epistemological Belief Questionnaire" to investigate the effects of epistemic beliefs and topic-specific beliefs on undergraduates' cognitive and strategic processing of a dual-positional text (thinking aloud).
Mason & Boscolo (2004)	They used an instrument by Kuhn et al. (2000) to measure the "epistemological understanding" by judging situations. The total scores could range from 15 (absolutist positions in all judgment domains) to 45 (evaluativist positions in all judgment domains).
Qian & Alvermann (1995)	They used the "Epistemological Belief Questionnaire" (Schommer, 1990) to correlate epistemic beliefs and learned helplessness with conceptual understanding and application reasoning in conceptual change learning.
Schommer, Calvert, Gariglietti, & Bajaj (1997)	They used Schommer's (1990) questionnaire.
Tsai (1998)	He used a questionnaire to identify 20 "information-rich" students that were then interviewed.

Table 2: An overview of the cited studies with a focus on methodology

Early studies in the 1970ies and 1980ies from developmental psychology modeled personal epistemology as a one-dimensional sequence of stages in which "individuals move through some specified sequence in their ideas about knowledge and knowing, as their ability to make meaning evolves." (Hofer, 2001, p. 356) Since the 1990ies, especially researchers from educational psychology consider personal epistemology as a system of more or less independent epistemic beliefs which allow for a more differentiated way of describing personal epistemology development as well as for a discipline-specific understanding of epistemology (cf. ibid., p. 361). A widely accepted structure for such a system of beliefs was proposed by Hofer and Pintrich (1997). According to Hofer and Pintrich, there are two general areas of epistemic beliefs (*nature of knowledge* and *nature or process of knowing*) with two dimensions each (see Table 1). A growing amount of psychological research presents relationships between epistemological beliefs and various aspects of learning. Examples (see Table 2 for these studies' methodological approaches) include the way in which college students' epistemological beliefs influence their processing of information and their monitoring of comprehension (e.g., Mason & Boscolo, 2004), their academic performance (e.g., Schommer, Calvert, Gariglietti, & Bajaj, 1997), conceptual change (e.g., Qian & Alvermann, 1995), cognitive processes during learning (e.g., Kardash & Howell, 2000), learning processes within computer-based scenarios and with the Internet (e.g., Hofer, 2004a), and their engagement in learning (e.g., Hofer & Pintrich, 1997). Furthermore, there is evidence that students' beliefs about knowledge and academic concepts depend on teaching style and epistemological beliefs of their teachers (e.g., Buelens, Clement, & Clarebout, 2002; Hofer, 2004b; Tsai, 1998). It is generally assumed that more sophisticated epistemological beliefs are related to more adequate learning strategies and therefore better learning outcomes (cf. Hofer & Pintrich, 1997; Stahl, 2011).

Trying to summarize recent research on epistemic cognition, one could say that most studies use self-report instruments like questionnaires to investigate correlations of epistemic beliefs with various aspects of learning and reception of information.

3 Certainty of knowledge

In this paragraph, we present the conceptualization of epistemic beliefs regarding the *certainty of knowledge* in different models of epistemic cognition as well as related research results. We further discuss the certainty of knowledge in a domain-specific view regarding mathematics drawing on the philosophy of mathematics.

3.1 Certainty of knowledge in models of epistemic cognition

Certainty of knowledge is a (sub-)dimension that is included in almost every conception of personal epistemology (cf. Hofer & Pintrich, 1997). In all of these models, a strong belief in truth and certainty is a sign of a naïve standpoint. For example, in Perry's (1970) original model of epistemic development, a *dualistic view* that sees knowledge claims either "right" or "wrong" is the least sophisticated stage. In higher stages of development, knowledge is considered to be constructed by humans and to be uncertain. In Schommer's (1990) questionnaire, strong beliefs in certainty of knowledge are considered as a sign of naivety. Hofer and Pintrich describe the dimension *certainty of knowledge* as follows:

> "The degree to which one sees knowledge as fixed or more fluid appears throughout the research, again with developmentalists likely to see this as a continuum that changes over time, moving from a fixed to a more fluid view. At lower levels, absolute truth exists with certainty. At higher levels, knowledge is tentative and evolving. […]" (Hofer & Pintrich, 1997, p. 119 f.)

Hofer and Pintrich (1997, p. 98) are consistent with Perry and other models of epistemic cognition in their description of this dimension: Absolutistic manifestations of this dimension go along with the view of "knowledge as certain and [the belief] that authorities have all the answers." As they become more sophisticated, learners recognize that knowledge may be uncertain and that authorities may possibly not know the truth. At even higher levels of sophistication, knowledge is regarded as "uncertain and contextual, but it is now possible to coordinate knowing and justification to draw conclusions across perspectives." (ibid, p. 101) Expert authority is critically evaluated at this stage.

Researchers slowly realize that epistemic beliefs are not general across all domains (cf. Hofer, 2006). However, the degree to which this conceptualization of certainty is applicable to domains like mathematics is yet to be evaluated and will be discussed in this paper.

The following is a summary of examples for research results using the conceptualization of certainty as illustrated above. In her first questionnaire study with 86 junior college students, Schommer (1990, p. 502 f.) identified a certainty dimension. "The more the students believed in certain knowledge, the more likely they were to write [inappropriately] absolute conclusions [in comprehension tasks]." In a study with 326 first-year college students, Hofer (2000, p. 402) identified a significant relation between the certainty dimension and academic achievement. "As expected, epistemological beliefs, at least on the dimension of certainty/ simplicity of knowledge, are correlated with academic performance. This was the case whether the dimension was discipline specific or general." Trautwein and Lüdtke (2007) conducted a study with 1094 students who were tested twice – in their final year of upper secondary school and two years later, after their first year at university. They showed that their certainty scale – even after controlling for other variables – was a significant predictor of the final school grade. Certainty beliefs also predicted the specific fields of study at university.

3.2 Certainty of mathematical knowledge

The *certainty of mathematical knowledge* differs from the certainty of knowledge in other disciplines and is therefore a deeply discussed topic in the philosophy of mathematics. There are good arguments for both – the certainty and the uncertainty of mathematical knowledge.

On the one hand, especially from a historical perspective, mathematical knowledge is regarded as certain, because of formal proofs and deductive reasoning with respect to strict rules and axioms (cf. Hoffmann, 2011, p. 1 ff.). This is a characteristic which is mostly unique for mathematics; compared to other (natural) sciences, mathematical knowledge does not depend on observations and experiments but only on logical conclusions. Arner (1972, p. 116) points out that "[t]he historical importance of mathematics as a paradigm of a priori truth needs no emphasis." According to Arner with the discovery of non-Euclidean geometries, all references to data and applications were abandoned.

> "Next it was demonstrated that not only geometry but other branches as well can be developed by the deductive method, from a relatively few assumptions, and likewise without reliance upon empirical data. As a result all pure mathematics is found to be abstract, in the sense of being independent of any particular application. […] It was here [in 'Principia Mathematica' by Whitehead and Russell] proved that the initial assumptions of mathematics can all be dispensed with, except the definitions. The truths of mathematics follow merely from definitions which exhibit the meaning of its concepts, by purely logical deduction. Judgment of such mathematical truth is, thus, completely and exclusively analytic; no synthetic judgment, a priori or otherwise, is requisite to knowledge of pure mathematics. The content of the subject consists entirely of the rigorous logical analysis of abstract concepts, in entire independence of all data of sense or modes of intuition." (Arner, 1972, p. 117 f.)

On the other hand, the assumption of absolute certainty revealed several flaws in the more recent history of mathematics:

Theoretical perspective: (1a) In formal proofs, each conclusion can be drawn by rules of inference relying on preceding sentences; this way, every mathematical statement can be traced back to very elementary rules and axioms. However, it is impossible to justify these axioms and the discovery of non-Euclidean geometries has shown that different stipulations of axioms can lead to divergent mathematics. Therefore, different prerequisites can lead to different outcomes concerning the same mathematical object. (1b) The finding of contradictory derivations from axioms (Russell's paradox, 1901) led to the attempt of establishing formal rules of derivation and to the effort of finding a complete and consistent set of axioms by Russell himself ("Principia Mathematica"), Hilbert and others. However, this attempt was regarded as a failing because of Gödel's incompleteness theorems in 1931 which stated that any sufficiently strong formal system cannot be both consistent and complete (cf. Hoffmann, 2011, p. 52 ff.)

Ontological perspective: (2) Mathematical knowledge cannot be objectively justified. There is no proof of a Platonian world to which mathematical results can be referred to.

Empirical perspective: (3a) Mathematical knowledge is spread by publishing in journals and to ensure the correctness of submitted proofs, mathematicians review each other's submissions. However, this review process cannot guarantee the identification of all inaccuracies or even incorrect parts that might be hidden in these proofs. Often, mathematical work is so specialized that only a handful of experts is able to comprehend it and the history of mathematics is full of examples of accepted proofs that were discovered to be wrong years after their publication (e.g., the proof of the Four Color Theorem, cf. Wilson, 2002). (3b) Finally, a growing number of mathematical results are achieved with the help of computers (e.g., the famous proof of the Four Color Theorem) and no mathematician is able to verify them without trusting the machines as well as hoping for error-free hard- and software (cf. Borwein & Devlin, 2011, p. 8 ff.).

The discussion about the fundamentals of mathematics during the last century has led to a panoply of different philosophical stances that coexist in the community of mathematicians and even within single persons, since there is no way of proving the validity of one position or another (e.g., Barrow, 1992; Bedürftig & Murawski, 2012). As a foundation for our research, this wealth of arguments is the basis for creating rich interview situations.

4 Methodology

In this paragraph, we explain our choice to conduct interviews and we describe the development and implementation of our interview setting. To assess epistemic beliefs about mathematics in depth, we chose different key questions from the philosophy of mathematics which provided us with a rich background of subject-specific theoretical positions and arguments. As an example of our research and as a central aspect of epistemic beliefs, we looked into the key question of *certainty of knowledge*.

4.1 Capturing epistemic beliefs: Methodological considerations

A common method of measuring epistemological beliefs is through the use of questionnaires. Very influential ones are the "Epistemological Questionnaire" (EQ) by Schommer (1990) and a questionnaire to measure teachers "views of mathematics and its structure" by Grigutsch, Raatz, and Törner (1998). Both are widely used among researchers and have inspired numerous variations and follow-up versions. However, questionnaires raise methodological issues regarding their validity and their effectiveness (cf. Stahl 2011; Muis 2004) as well as psychometric problems that all self-report instruments suffer from (Greene & Yu 2004).

Therefore, both Stahl and Greene and Yu suggest the use of interviews, observations, or other instruments rather than classical questionnaires to understand epistemic beliefs of learners in detail. Following these recommendations in the study presented here, we use interviews to measure domain-specific (mathematics) epistemic beliefs.

4.2 Development and implementation of the interviews

To initiate the interviews, we presented quotes of representatives of opposing epistemic positions to the interviewees and asked them to relate themselves to these statements. This proved to generate more insight into our subjects' beliefs than direct questioning (e.g., "Do you think that mathematical knowledge is certain?"), because this way, our interviewees had different positions to refer to and got an impression of the breadth of the argument. After this initial prompt, we asked further questions and intervened with information contrary to the subjects' positions to further identify their lines of reasoning (see below). Presenting specific contexts and thus specifying the type of knowledge that is discussed is vital to guarantee measurement validity (cf. Greene & Yu, 2014, p. 23).

During the first phase of data collection, we optimized our selection of quotes to start the interviews with as well as the pursuing questions. For example, we added explicit headlines to our quotes ("mathematical knowledge is certain/ uncertain", see Table 3) instead of solely presenting the quotes to emphasize the two positions. Starting with only one intervention for all students ("is mathematical knowledge really certain?"), we also developed additional interventions for the various positions and reasons our interviewees raised. The data collection ended when the authors felt that the data were saturated. From a methodological point of view, the development of the research design with respect to successively analyzed data and the collection of data until saturation is reached, is similar to the approach of Grounded Theory (cf. Strauss & Corbin, 1996).

So far, the first author interviewed 10 pre-service teachers of mathematics (students at the University of Education Freiburg), 2 in-service teachers of mathematics, 2 professional mathematicians and 2 professors of mathematics as well as 1 professor of economics and 1 PhD student from veterinary medicine. The interview data has been analyzed and validated consensually within intense discussions among the researchers.

In our sample setting, the interview started like this: "These are two positions of mathematicians regarding the certainty of mathematical knowledge. With which position can you identify yourself? Please give reasons for your answer." (See Table 3 for the two positions.) Further questions were: "Can you explain your position on the basis of your own mathematical experience?" and "Please compare the certainty of mathematical knowledge to that of other sciences, for example to physical, linguistic, or educational knowledge." These questions proved to be helpful in revealing our interviewees' content knowledge, methodological knowledge, and ontological assumptions.

Mathematical Knowledge is Certain	Mathematical Knowledge is Uncertain
"In mathematics, knowledge is valid forever. A theorem is never incorrect. In contrast to all other sciences, knowledge is accumulated in mathematics. […] It is impossible, that a theorem that was proven correctly will be wrong from a future point of view. Each theorem is for eternity." (Albrecht Beutelspacher) [2001, p. 235; translated by the first author]	"The issue is […] whether mathematicians can always be absolutely confident of the truth of certain complex mathematical results […]. With regard to some very complex issues, truth in mathematics is that for which the vast majority of the community believes it has compelling arguments. And such truth may be fallible. Serious mistakes are relatively rare, of course." (Alan H. Schoenfeld) [1994, p. 58 f.]

Table 3: Starting positions for "Certainty of Mathematical Knowledge".

If an interviewee settled on "mathematical knowledge is *certain*", we confronted him/her with information about the scientific review process and the history of the Four Color Theorem (cf. Wilson, 2002): In 1879, Kempe submitted a proof for this problem that was accepted by the community of mathematicians, only to be shown to be false by Heawood eleven years later. The currently valid proof by Apple and Haken is so complicated and so large (more than 400 pages) that no one can be sure of its correctness.

If a subject expressed that "mathematical knowledge is *uncertain*", we gave him/her additional information about deductive reasoning and presented an easy proof of the Pythagorean Theorem. "How can a theorem like this one, with hundreds of proofs, countless validations and practical applications be regarded as uncertain?"

Overall, applying all the questions and prompts described here as well as questions that arose from the situation, the interviews lasted between 5 and 20 minutes.

5 Results

After interviewing university students of mathematics as well as professional mathematicians and professors of mathematics, we can present the following stances regarding the certainty of mathematical knowledge and underpin them with supporting statements.[1]

[1] The interviews have all been conducted in German; quotes have been translated by the first author.

We found several representatives of either the position that "mathematical knowledge is certain" or that "mathematical knowledge is uncertain" as well as some interviewees that were undecided. For both "certain" and "uncertain" we found participants that showed substantially different argumentations due to their backgrounds. In the following we present three examples for each position showing graduated levels of sophistication.

5.1 Interviewees judging that "mathematical knowledge is certain"

T.W. is a pre-service mathematics teacher in his third semester (age 23). For him, mathematical knowledge is certain, "the first quote of Beutelspacher is more likely correct in my view." He thinks of proofs as inevitable and irrefutable and adds: "How can there possibly be errors in mathematics?" Confronted with the historical episode of the four color theorem, T.W. admits "Of course, there can be errors, […] but it got proven eventually, didn't it?" He mentioned the Pythagorean Theorem as an example of an everlasting theorem and did not know about the mathematical review process (and its limitations) before our intervention.

P.S. is a doctoral student in veterinary medicine in her second year (age 28). She shows a very sophisticated knowledge of the way scientific results are gained and justified in the natural sciences like biology and medicine. But she has no concrete idea of the way mathematical results are obtained. Asked to compare the certainty of mathematical knowledge to that of other scientific disciplines, P.S. answers: "I'd say it is more certain than biological knowledge. Very similar to that of physical knowledge, because both, mathematics and physics, […] build on ideas and logical thinking but less on tangible things." Compared to medical research, she imagines mathematical results to be not as easily refuted because experiments cannot prove them wrong.

A.R. is a mathematician who just obtained his diploma degree (age 31). He was able to activate more content knowledge than the previous two cited interviewees. For example, he referred to Andrew Wiles' proof of Fermat's Last Theorem as well as the Riemann Conjecture. His claim of certainty was based on the deductive way of reasoning in mathematics. Even though he knew about possible flaws in the review process, A.R. did attribute this error not to mathematics: "Humans are fallible. […] There might be errors in proofs which are accepted by many people. […] But when a theorem is proved correctly from the axioms by formal rules of derivation then it will last for eternity."

Interpretation: The three interviewees above all represent the position that mathematical knowledge is certain, but the degree of sophistication varies significantly. T.W. has naïve beliefs of certainty; for him, there is not a real possibility for doubt in mathematical knowledge. P.S. is an expert in the process of knowledge generation in the natural sciences, but she has no knowledge about mathematical research.

She cannot argue with possible flaws of mathematic-specific results as she can do with results from biological research. A.R. does not only know about the scientific review process and its flaws, but also about the specific way of reasoning in mathematics and about examples from the history of mathematics. Of those three, A.R. represents the most sophisticated epistemic beliefs regarding mathematical knowledge.

5.2 Interviewees judging that "mathematical knowledge is uncertain"

B.G. is a pre-service mathematics teacher who just finished her university studies (age 25). She thinks that not only mathematical but all knowledge is uncertain, because "for me, there is always the possibility that someone figures out that something is not quite correct. A theorem might be proven and checked but there is always the possibility of finding an aspect that it may not be correct." In comparison to other scientific disciplines, she thinks that mathematical knowledge is quite certain; but general uncertainty remains: "I generally do not agree to statements referring to 'ever' or 'never'."

C.P. is a pre-service mathematics teacher in her fourth semester (age 23). She thinks that mathematical knowledge is uncertain, "because it has occurred several times in the history of mathematics that a theorem or its proof has proven to be false." After further questions she admits that all mathematical results taught at school are beyond doubt but there can always be errors in complex results at universities and in mathematical research.

S.W. is a professor of mathematics (age 41). He used arguments referring to the review process as well as ontological positions: "Mathematical knowledge cannot be definitely certain because that would imply an infallible system of rules with an otherworldly justification. Mathematics would need a justification outside of the human sphere and outside of the mathematical discourse, a realm that could be observed and described. That there is such a realm, such a sphere, I am very skeptical about."

Interpretation: These three interviewees all claimed that mathematical knowledge is uncertain. Whereas B.G. could only rely on fundamental conceptions ("nothing is certain"), C.P. was able to support her position with knowledge about the review process and events from the history of mathematics. S.W. is the most sophisticated representative of these three. He does not only use arguments regarding the review process and historical events, but he also refers to the ontology of mathematical knowledge.

6 Discussion and outlook

Certainty of Knowledge is an important category in nearly all models of epistemic cognition. It is generally assumed that beliefs of certainty are a sign for absolutistic or unreflected beliefs whereas beliefs of uncertainty hint at unsophisticated beliefs about the nature of knowledge and knowing. However, there are serious doubts regarding the domain-generality of such assumptions (cf. Hofer, 2006).

Drawing on domain-specific beliefs concerning the nature of mathematical knowledge, we conducted an interview study with students as well as professors of mathematics. The analyses of the interviews reveal a breadth of arguments with respect to the epistemological question whether mathematical knowledge is certain or uncertain. We encountered unreflected as well as reflected representatives of both statements. This phenomenon contrasts the assumption from research on personal epistemology that "absolute truth exists with certainty" is valid only for "lower levels" of sophistication (Hofer & Pintrich, 1997, p. 119 f.). As can be seen by our interview data (especially by the example of A.R.), this belief in the certainty of mathematical knowledge can be held in a reflected way. This is in line with recent findings of Greene and Yu (2014, p. 18) who concluded: "[…] believing 'knowledge,' as a general concept, is 'certain' is not a reliable indicator of naivety." On the other hand, there are interviewees that believe in the uncertainty of mathematical knowledge but do not use sophisticated arguments for their position. Again, referring to Greene and Yu (ibid., p. 25), "merely disagreeing with a naïve statement is not a reliable indicator of expertise".

Building on the analyses of the interview data reported in this article, we developed an instrument with the aim to identify epistemic beliefs about mathematics effectively and reliably. Because of the known problems with common, paper-based questionnaires (see section 4.1, on "Methodological Issues"), we settled for a web-based questionnaire that is able to adapt to the participants' responses. If, for example, a participant answers "mathematical knowledge is certain", he will get the "Four Color Theorem" intervention but not the "Pythagorean Theorem" intervention (see "Development and Implementation of the Interviews"). A first study using this questionnaire with about 150 pre-service teachers of mathematics is reported in Rott, Leuders, and Stahl (submitted *a*).

It may also be asked whether the phenomena reported with respect to the certainty or uncertainty of mathematical knowledge also occur in other topics regarding the epistemic or ontological quality of mathematics. Indeed similar findings can be reported regarding the topic of the *justification of mathematical knowledge,* which is discussed elsewhere (Rott, Leuders & Stahl submitted *b*)

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